# Comparison Theorems of Spectral Gaps of Schrödinger Operators and Diffusion Operators

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# Outline

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1. Spectral Gap of Semigroups
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   - Log-concavity estimate of ground state
   - Probabilistic Proof of the Conjecture
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# Feynman-Kac semigroup

Let $(\Omega, (X_t)_{t\in\mathbb{R}^+}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, (\mathbb{P}_x)_{x\in E})$ be a càdlàg Markov process on the state space $E$. Assume the transition Markov semigroup $P_t$ is symmetric in some $L^2(\mu)$ and essentially irreducible.

Given a potential $V : E \to \mathbb{R}$, define the Feynman-Kac semigroup:

$$P_t^V f(x) \ := \ \mathbb{E}^x f(X_t) \exp\left(\int_0^t V(X_s)ds\right), \ \forall f \geqslant 0.$$

Let $-\mathcal{L}^V$ be the lower-bounded self-adjoint Schrödinger operator generated by $P_t^V$. Define the lowest spectral point

$$\lambda_0(V) \ = \ \inf\left\{\int Vf^2 \mathrm{d}\mu + \int f\cdot(-\mathcal{L}^V f)\mathrm{d}\mu; \right.$$
$$\left. f \in D(\mathcal{L}^V) \cap L^2(V^+\mu), \int f^2 \mathrm{d}\mu = 1\right\}.$$

People are concerned with:

- When is $\lambda_0(V)$ isolated in the spectrum $\sigma(-\mathcal{L}^V)$?
- How to characterize the ground state $\phi_0$ corresponding to $\lambda_0(V)$? For example, is $\phi_0$ strictly positive and "concave"?
- How to estimate the gap between $\lambda_0(V)$ and the bottom of essential spectrum of $-\mathcal{L}^V$? Equivalently, how to estimate the exponential convergence rate of the Markov process under Girsanov transformation corresponding to ground state to its stability state?
- Logarithmic Sobolev inequality with respect to $\phi_0^2 \mathrm{d}\mu$....

# Girsanov semigroup

As a counterpart, we can also consider a Girsanov semigroup as follows. Assume further that $X_t$ is conservative.

Let $\nu \ll \mu$, and $(L_t)_{t\geqslant 0}$ is an additive $\mathbb{P}_\mu$-local martingale associated with $\nu$. Define a perturbation of $\mathbb{P}_\mu$ by the Girsanov's formula:

$$\begin{aligned} Q_{\nu|\mathcal{F}_t} &:= \exp\left(L_t - \frac{1}{2}\langle L\rangle_t\right)\mathbb{P}_{\mu|\mathcal{F}_t}, \\ Q_t f(x) &:= \mathbb{E}^{\mathbb{P}_\nu}\left[f(X_t)\exp\left(L_t - \frac{1}{2}\langle L\rangle_t\right)\Big|X_0 = x\right]. \end{aligned}$$

The conversation implies that 0 is just the lowest eigenvalue for the generator of $Q_t$. We can ask the same questions as previous.

# Existence

Indeed, we have found some criterions to yield the existence of the spectral gap, see

Fuzhou Gong, Liming Wu, *Spectral gap of positive operators and applications*. J. Math. Pures Appl. 85 (2006), 151–191.

For simplicity, here we just give an application to abstract Wiener space $(\mathbb{W}, \mathbb{H}, \mu)$ endowed with the Ornstein–Uhlenbeck operator $\mathcal{L}$.

Given $b : \mathbb{W} \to \mathbb{H}$, especially $b = \nabla V$, consider the Girsanov semigroup associated to the diffusion operator $\mathcal{L}_b := \mathcal{L} + b \cdot \nabla$. We have:

**Theorem 1**

*If for some* $\lambda > 1$ *holds*

$$\int \exp(\lambda \cdot |b|_{\mathbb{H}}^2) \mathrm{d}\mu < +\infty,$$

*then* $\mathcal{L}_b$ *has a spectral gap in* $L^p(\mu)$ *for any* $p > 1$.

Note that, the above integrability condition is sharp. However, there was no nice estimates on the spectral gap or ground state.

Roughly speaking, maybe some control on the "derivative" of $b$ is needed at least outside some bounded domain, otherwise a high-frequency vibration on $b$ outside a bounded domain maybe impact heavily on the scale of spectral gap, but make no difference to the integrability. Hence, in the first step one consider the case that, there is a global control on the "derivative" of $b = \nabla V$.

# Outline

Some notations:

- $\Omega \subset \mathbb{R}^n$: a bounded convex domain of diameter $D = \mathrm{diam}(\Omega)$;
- $V : \Omega \to \mathbb{R}$ a convex potential;
- $L = -\Delta + V$: the Schrödinger operator on $\Omega$ with Dirichlet boundary condition;
- Eigenvalues of $L$: $\lambda_0 < \lambda_1 \le \lambda_2 \le \ldots$, $\lim_{i\to\infty} \lambda_i = +\infty$;
- Eigenfunctions of $L$: $\phi_0, \phi_1, \phi_2, \ldots$, $\phi_i|_{\partial\Omega} \equiv 0$.

$\phi_0$ and $\lambda_0$ are called the ground state and ground state energy, respectively. $\phi_0$ is strictly positive in $\Omega$.

**Gap Conjecture** (van den Berg, 1983): The spectral gap of $L$ satisfies

$$\lambda_1 - \lambda_0 \ge \frac{3\pi^2}{D^2}. \tag{1}$$

## Example 2

Consider the one dimensional case $\Omega = (-\frac{D}{2}, \frac{D}{2}) \subset \mathbb{R}^1$ and $V \equiv 0$. Then the operator is given by $L = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}$, and

| | Eigenvalues $\lambda_i$ | Eigenfunctions $\phi_i$ |
|---|---|---|
| $i = 0$ | $\frac{\pi^2}{D^2}$ | $\cos \frac{\pi t}{D}$ |
| $i = 1$ | $\frac{4\pi^2}{D^2}$ | $\sin \frac{2\pi t}{D}$ |

Therefore the spectral gap is $\frac{3\pi^2}{D^2}$.

# Known results

In one dimension:

- Ashbaugh & Benguria (1989): If $V$ is symmetric and single-well (not necessarily convex), then the conjecture holds;
- Lavine (1994): The conjecture holds if $V$ is convex.

In higher dimensions:

- Singer, Wong, Yau & Yau (1985): The gap $\lambda_1 - \lambda_0 \geq \frac{\pi^2}{4D^2}$;
- Qi Huang Yu & Jia Qing Zhong (1986): The gap $\lambda_1 - \lambda_0 \geq \frac{\pi^2}{D^2}$;
- $\ldots$;
- Andrews & Clutterbuck (2011): The gap conjecture holds. **Basic idea**: compare the spectral gap with one dimensional case.

# Modulus of convexity

Let $\tilde{V} \in C^1([-\frac{D}{2}, \frac{D}{2}], \mathbb{R})$ be an even function, such that $\forall x, y \in \Omega, x \neq y$,

$$\left\langle \nabla V(x) - \nabla V(y), \frac{x-y}{|x-y|} \right\rangle \geq 2\tilde{V}'\left(\frac{|x-y|}{2}\right). \qquad (2)$$

The function $\tilde{V}$ is called a modulus of convexity of $V$.

**Remark 3**

*(i) If the sign $\geq$ is replaced by $\leq$, then $\tilde{V}$ is called a modulus of concavity of $V$.*

*(ii) If $V$ is convex, then we can choose $\tilde{V} \equiv 0$.*

# Log-concavity estimate of ground state

Consider the one dimensional Schrödinger operator $\tilde{L} = -\frac{d^2}{dt^2} + \tilde{V}$ on the symmetric interval $[-\frac{D}{2}, \frac{D}{2}]$, satisfying the Dirichlet boundary condition.

Denote by the corresponding objects by adding a tilde, e.g. $\tilde{\lambda}_i$ and $\tilde{\phi}_i$, $i = 0, 1, 2, \ldots$.

**Theorem 4 (Andrews & Clutterbuck, JAMS, 2011, Theorem 1.5)**

*Assume that $\tilde{V}$ is a modulus of convexity of $V$, i.e. (2) holds, then $\log \tilde{\phi}_0$ is a modulus of concavity of $\log \phi_0$. More precisely, $\forall\, x, y \in \Omega, x \neq y$,*

$$\left\langle \nabla \log \phi_0(x) - \nabla \log \phi_0(y), \frac{x-y}{|x-y|} \right\rangle \le 2(\log \tilde{\phi}_0)' \left( \frac{|x-y|}{2} \right). \quad (3)$$

# Remarks on Theorem 4

## Remark 5

- *Recall that when* $V$ *is convex, then* $\tilde V \equiv 0$.

  *In this case,* $\tilde L = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}$ *has the ground state* $\tilde\phi_0(t) = \cos\frac{\pi t}{D}$, *thus* $(\log\tilde\phi_0)'(t) = -\frac{\pi}{D}\tan\frac{\pi t}{D}$, $t \in (-\frac{D}{2}, \frac{D}{2})$.

  *The log-concavity estimate* (3) *becomes*

$$\left\langle \nabla\log\phi_0(x) - \nabla\log\phi_0(y), \frac{x-y}{|x-y|}\right\rangle \le -\frac{2\pi}{D}\tan\left(\frac{|x-y|}{2D}\right). \tag{4}$$

- *Brascamp & Lieb (JFA, 1976) proved a weaker result: if* $V$ *is convex, then the ground state* $\phi_0$ *is log-concave.*

# Spectral gap comparison theorem

**Theorem 6 (Andrews & Clutterbuck, JAMS, 2011, Theorem 1.3)**

*If $\tilde{V}$ is a modulus of convexity of $V$, i.e.* (2) *holds, then* $\lambda_1 - \lambda_0 \geq \tilde{\lambda}_1 - \tilde{\lambda}_0$.

Ingredients of the proof:

(i) the ground state transform: let $u_i(t,x) = e^{-\lambda_i t}\phi_i(x)$ and $v = \frac{u_1}{u_0} = e^{-(\lambda_1-\lambda_0)t}\frac{\phi_1}{\phi_0}$, then $v(t,\cdot) \in C^\infty(\bar{\Omega})$ and

$$\frac{\partial v}{\partial t} = \Delta v + 2\nabla \log \phi_0 \cdot \nabla v;$$

(ii) sharp log-concavity estimate of ground state $\phi_0$ (Theorem 4);

(iii) estimate of the modulus of continuity:

$$v(t,x) - v(t,y) \le C\tilde{v}(t,|x-y|) = Ce^{-(\tilde{\lambda}_1-\tilde{\lambda}_0)t}\frac{\tilde{\phi}_1}{\tilde{\phi}_0}(|x-y|).$$

Recall that $v(t,x) - v(t,y) = e^{-(\lambda_1-\lambda_0)t}\left(\frac{\phi_1}{\phi_0}(x) - \frac{\phi_1}{\phi_0}(y)\right)$, hence $\forall\, t \ge 0$ and $x, y \in \Omega$,

$$e^{-(\lambda_1-\lambda_0)t}\left(\frac{\phi_1}{\phi_0}(x) - \frac{\phi_1}{\phi_0}(y)\right) \le Ce^{-(\tilde{\lambda}_1-\tilde{\lambda}_0)t}\frac{\tilde{\phi}_1}{\tilde{\phi}_0}(|x-y|)$$

which implies $\lambda_1 - \lambda_0 \ge \tilde{\lambda}_1 - \tilde{\lambda}_0$.

**Our purpose**: give a probabilistic proof to the gap conjecture by using the coupling by reflection.

## Outline

# Equations for $\log\phi_0$

In order to estimate the log-concavity of $\phi_0$, we observe that $-\Delta\phi_0 + V\phi_0 = \lambda_0\phi_0$. Hence

$$\Delta \log \phi_0 + |\nabla \log \phi_0|^2 = V - \lambda_0.$$

Differentiating the equation leads to

$$\Delta(\nabla \log \phi_0) + 2\big\langle \nabla \log \phi_0, \nabla(\nabla \log \phi_0)\big\rangle = \nabla V, \tag{5}$$

or equivalently, in component form,

$$\Delta(\partial_i \log \phi_0) + 2\big\langle \nabla \log \phi_0, \nabla(\partial_i \log \phi_0)\big\rangle = \partial_i V, \quad 1 \le i \le n.$$

# Conservative diffusion

The above equations suggest us to consider the following SDE

$$\mathrm{d}X_t = \sqrt{2}\,\mathrm{d}B_t + 2\nabla \log \phi_0(X_t)\,\mathrm{d}t, \quad X_0 = x \in \Omega. \tag{6}$$

where $B_t$ is an $n$-dimensional standard Brownian motion.

The diffusion $(X_t)_{t\geq 0}$ is conservative, that is, starting from a point $x \in \Omega$, the process $X_t$ will not arrive at the boundary $\partial\Omega$.

- Eric Carlen (Commun. Math. Phys., 1984), P.A. Meyer & W.A. Zheng (Séminaire de probabilités, 1985);
- We can also consider the one dimensional process $\rho_{\partial\Omega}(X_t)$, where $\rho_{\partial\Omega} : \Omega \to \mathbb{R}_+$ is the distance function to the boundary. Using the properties of the drift $2\nabla \log \phi_0 = 2\frac{\nabla\phi_0}{\phi_0}$, we can prove $\rho_{\partial\Omega}(X_t) > 0$ a.s. $\forall\, t \geq 0$.

# Coupling by reflection

To introduce the coupling by reflection of $(X_t)_{t\geq 0}$, we define

$$M(x,y) = I_n - 2\frac{(x-y)(x-y)^*}{|x-y|^2}, \quad x, y \in \mathbb{R}^n, x \neq y,$$

which is the matrix of the reflection mapping w.r.t. the hyperplane passing through $\frac{x+y}{2}$ and perpendicular to the vector $x - y$.

Consider

$$\mathrm{d}Y_t = \sqrt{2}\, M(X_t, Y_t)\,\mathrm{d}B_t + 2\nabla \log \phi_0(Y_t)\,\mathrm{d}t, \quad Y_0 = y \in \Omega. \quad (7)$$

The coupled process $(X_t, Y_t)_{t\geq 0}$ is called the coupling by reflection.

## Some notations

For $\eta, \delta > 0$, define stopping times

$$\tau_\eta = \inf\{t > 0 : |X_t - Y_t| = \eta\},$$
$$\sigma_\delta = \inf\{t > 0 : \rho_{\partial\Omega}(X_t) \wedge \rho_{\partial\Omega}(Y_t) = \delta\}.$$

In view of the log-concavity estimate (4), we consider the processes

$$\alpha_t = \nabla \log \phi_0(X_t) - \nabla \log \phi_0(Y_t),$$
$$\beta_t = \frac{X_t - Y_t}{|X_t - Y_t|},$$
$$F_t = \langle \alpha_t, \beta_t \rangle.$$

Then $\quad d(X_t - Y_t) = 2\sqrt{2}\,\beta_t\langle\beta_t, dB_t\rangle + 2\alpha_t\,dt \quad$ and

$$F_0 = \left\langle \nabla \log \phi_0(x) - \nabla \log \phi_0(y), \frac{x-y}{|x-y|} \right\rangle.$$

# Two lemmas

## Lemma 7

*Assume that the potential* $V : \bar{\Omega} \to \mathbb{R}$ *is convex. Then for* $t \leq \tau_\eta \wedge \sigma_\delta$,

$$\mathrm{d}F_t \geq \langle \beta_t, \mathrm{d}M_t \rangle, \tag{8}$$

*where*

$$M_t = \sqrt{2} \int_0^t \left[ (\nabla^2 \log \phi_0)(X_s) - (\nabla^2 \log \phi_0)(Y_s) M(X_s, Y_s) \right] \mathrm{d}B_s.$$

The proof follows from Itô's formula, the equation (5) for $\log \phi_0$ and the property of coupling by reflection:

$$\mathrm{d}|X_t - Y_t| = \left\langle \frac{X_t - Y_t}{|X_t - Y_t|}, \mathrm{d}(X_t - Y_t) \right\rangle = 2\sqrt{2}\, \langle \beta_t, \mathrm{d}B_t \rangle + 2F_t\, \mathrm{d}t.$$

# Two lemmas

Let $\tilde\phi_{D,0}(t) = \cos\frac{\pi t}{D}$, $t \in [-\frac{D}{2}, \frac{D}{2}]$ be the first Dirichlet eigenfunction of the operator $-\frac{\mathrm{d}^2}{\mathrm{d}t^2}$ on the interval $[-\frac{D}{2}, \frac{D}{2}]$.

$\psi_D(t) = (\log\tilde\phi_{D,0})'(t) = -\frac{\pi}{D}\tan\frac{\pi t}{D}$, $t \in (-\frac{D}{2}, \frac{D}{2})$.

Since $\psi_D(t)$ explodes at $t = \pm\frac{D}{2}$, we take $D_1 > D$ and consider $\tilde\phi_{D_1,0}$, $\psi_{D_1}$. Then $\psi_{D_1} \in C_b^\infty[0, \frac{D}{2}]$ and it satisfies

$$\psi''_{D_1} + 2\psi_{D_1}\psi'_{D_1} = 0.$$

**Lemma 8**

*Set $\xi_t = |X_t - Y_t|/2$. We have for $t \le \tau_\eta \wedge \sigma_\delta$,*

$$\mathrm{d}\psi_{D_1}(\xi_t) = \sqrt{2}\,\psi'_{D_1}(\xi_t)\langle\beta_t, \mathrm{d}B_t\rangle + \psi'_{D_1}(\xi_t)\big[F_t - 2\psi_{D_1}(\xi_t)\big]\,\mathrm{d}t.$$

# Log-concavity estimate of the ground state

**Theorem 9 (Modulus of log-concavity)**

*Assume that the potential function $V : \Omega \to \mathbb{R}$ is convex. Then for all $x, y \in \Omega$ with $x \neq y$, it holds*

$$\left\langle \nabla \log \phi_0(x) - \nabla \log \phi_0(y), \frac{x-y}{|x-y|} \right\rangle \leq -\frac{2\pi}{D} \tan\left(\frac{\pi|x-y|}{2D}\right).$$

Sketch of proof. Fix $\eta > 0$, $\delta > 0$ and $D_1 > D$. Lemmas 7 and 8 lead to

$$\mathrm{d}\big[F_t - 2\psi_{D_1}(\xi_t)\big] \geq \mathrm{d}\tilde{M}_t - 2\psi'_{D_1}(\xi_t)\big[F_t - 2\psi_{D_1}(\xi_t)\big]\,\mathrm{d}t,$$

in which $\mathrm{d}\tilde{M}_t$ is the martingale part.

# Sketch of proof

The above inequality is equivalent to

$$\mathrm{d}\left(\left[F_t - 2\psi_{D_1}(\xi_t)\right] \exp\left[\int_0^t 2\psi'_{D_1}(\xi_s)\,\mathrm{d}s\right]\right) \geq \exp\left[\int_0^t 2\psi'_{D_1}(\xi_s)\,\mathrm{d}s\right] \mathrm{d}\tilde{M}_t.$$

Integrating from 0 to $t \wedge \tau_\eta \wedge \sigma_\delta$ and taking expectation on both sides give us

$$\begin{aligned} &F_0 - 2\psi_{D_1}(\xi_0) \\ &\leq \mathbb{E}\left(\left[F_{t\wedge\tau_\eta\wedge\sigma_\delta} - 2\psi_{D_1}(\xi_{t\wedge\tau_\eta\wedge\sigma_\delta})\right] \exp\left[\int_0^{t\wedge\tau_\eta\wedge\sigma_\delta} 2\psi'_{D_1}(\xi_s)\,\mathrm{d}s\right]\right). \end{aligned} \tag{9}$$

Brascamp & Lieb (JFA, 1976): if $V$ is convex, then the ground state $\phi_0$ is log-concave. Hence $F_{t\wedge\tau_\eta\wedge\sigma_\delta} \leq 0$ a.s.

# Sketch of proof

$$F_0 - 2\psi_{D_1}(\xi_0) \le -2\,\mathbb{E}\left(\psi_{D_1}(\xi_{t\wedge\tau_\eta\wedge\sigma_\delta})\exp\left[\int_0^{t\wedge\tau_\eta\wedge\sigma_\delta} 2\psi'_{D_1}(\xi_s)\,\mathrm{d}s\right]\right).$$

Lindvall & Rogers (Ann. Probab., 1986): the log-concavity of the drift $\nabla\log\phi_0$ implies the coupling $(X_t, Y_t)$ is successful, i.e., $\tau_\eta \uparrow \tau < +\infty$ a.s.

Note that $\psi_{D_1}$ is a bounded function on $[0, D/2]$.
Moreover, $\psi'_{D_1}(z) = -\frac{\pi^2}{D_1^2}\sec^2\left(\frac{\pi z}{D_1}\right) \le 0$ for $z \in [0, D/2]$, thus $\exp\left[\int_0^{t\wedge\tau_\eta\wedge\sigma_\delta} 2\psi'_{D_1}(\xi_s)\,\mathrm{d}s\right] \le 1$ for all $t > 0$.

Letting $t \uparrow \infty$ and $\delta, \eta \downarrow 0$, the dominated convergence theorem yields

$$F_0 - 2\psi_{D_1}(\xi_0) \le -2\,\mathbb{E}\left(\psi_{D_1}(\xi_\tau)\exp\left[\int_0^{\tau} 2\psi'_{D_1}(\xi_s)\,\mathrm{d}s\right]\right) = 0.$$

If we do not use the results of Brascamp & Lieb (JFA, 1976) and Lindvall & Rogers (Ann. Probab., 1986), then we need two more estimates on the ground state $\phi_0$:

- the first one concerns the near diagonal behavior of $\nabla \log \phi_0$;
- the second one is the asymptotics of $\nabla \log \phi_0$ near the boundary $\partial\Omega$.

**Lemma 10 (Near-diagonal estimate)**

*For any $\varepsilon > 0$, there is $\eta_1 > 0$ such that for all $x, y \in \Omega$ with $|x - y| \le \eta_1$, it holds*

$$\left\langle \nabla \log \phi_0(x) - \nabla \log \phi_0(y), \frac{x-y}{|x-y|} \right\rangle \le \varepsilon.$$

## Lemma 11 (Near-boundary estimate)

*Let $\eta_1 > 0$ be given as in Lemma 10. There is $\delta_1 > 0$ small enough such that if $\delta < \delta_1$ and $x \in \partial_\delta \Omega$, $y \in \Omega$ with $|x - y| > \eta_1$, it holds*

$$\left\langle \nabla \log \phi_0(x) - \nabla \log \phi_0(y), \frac{x-y}{|x-y|} \right\rangle \le -C_1 \log \frac{\delta_1}{\delta} + C_2$$

*for some constants $C_1, C_2 > 0$.*

Now choose any $\varepsilon > 0$. Letting $t \uparrow \infty$ in (9) yields

$$\begin{aligned} & F_0 - 2\psi_{D_1}(\xi_0) \\ & \le \mathbb{E}\left( \left[ F_{\tau_{\eta_1} \wedge \sigma_{\delta_2}} - 2\psi_{D_1}(\xi_{\tau_{\eta_1} \wedge \sigma_{\delta_2}}) \right] \exp \left[ \int_0^{\tau_{\eta_1} \wedge \sigma_{\delta_2}} 2\psi'_{D_1}(\xi_s) \, ds \right] \right) \end{aligned}$$

for sufficiently small $\delta_2 < \delta_1$. By Lemmas 10 and 11, we can prove

$$F_{\tau_{\eta_1} \wedge \sigma_{\delta_2}} - 2\psi_{D_1}(\xi_{\tau_{\eta_1} \wedge \sigma_{\delta_2}}) \le 2\varepsilon.$$

# Outline

# Simple preparation

Recall the processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ defined in (6) and (7). We still denote by $\xi_t = |X_t - Y_t|/2$ which satisfies

$$\mathrm{d}\xi_t = \sqrt{2}\,\langle \beta_t, \mathrm{d}B_t\rangle + F_t\,\mathrm{d}t \leq \sqrt{2}\,\langle \beta_t, \mathrm{d}B_t\rangle - \frac{2\pi}{D}\tan\left(\frac{\pi\xi_t}{D}\right)\mathrm{d}t.$$

**Lemma 12**

*We have for all $t \geq 0$,*

$$\mathbb{E}\sin\left(\frac{\pi\xi_t}{D}\right) \leq \exp\left(-\frac{3\pi^2 t}{D^2}\right)\sin\left(\frac{\pi|x-y|}{2D}\right) \leq \exp\left(-\frac{3\pi^2 t}{D^2}\right).$$

Proof of the gap conjecture (1). Recall the ground state transform $v = \frac{e^{-\lambda_1 t}\phi_1}{e^{-\lambda_0 t}\phi_0} =: e^{-(\lambda_1-\lambda_0)t} v_0$ solves

$$\frac{\partial v}{\partial t} = \Delta v + 2\nabla \log \phi_0 \cdot \nabla v.$$

Hence $\qquad v(t,x) = \mathbb{E} v_0(X_t), \quad v(t,y) = \mathbb{E} v_0(Y_t).$

Since $v_0 = \frac{\phi_1}{\phi_0}$ is Lipschitz continuous on $\bar{\Omega}$ with a constant $K > 0$,

$$|v(t,x) - v(t,y)| \le \mathbb{E}|v_0(X_t) - v_0(Y_t)| \le K\mathbb{E}|X_t - Y_t| = 2K\mathbb{E}\xi_t.$$

Next $\sin\frac{\pi z}{D} \ge \frac{2z}{D}$ for $z \in [0, \frac{D}{2}]$, hence

$$|v(t,x) - v(t,y)| \le KD\mathbb{E}\sin\left(\frac{\pi \xi_t}{D}\right) \le KD\exp\left(-\frac{3\pi^2 t}{D^2}\right),$$

where the last inequality is due to Lemma 12.

Noting that $v(t,x) - v(t,y) = e^{-(\lambda_1 - \lambda_0)t}(v_0(x) - v_0(y))$, we obtain

$$e^{-(\lambda_1-\lambda_0)t}|v_0(x) - v_0(y)| \le KD \exp\left(-\frac{3\pi^2 t}{D^2}\right)$$

for all $t \ge 0$ and $x, y, \in \Omega$. Since $v_0 = \frac{\phi_1}{\phi_0}$ is not constant, we conclude that

$$\lambda_1 - \lambda_0 \ge \frac{3\pi^2}{D^2}.$$

# Outline

# From Euclid to Wiener

In our opinion, the "modulus of convexity" performs uniformly as a lower bound of Hessian($V$) in each direction and each interval.

And the most interesting thing is, this kind of control will be inherited by the logarithm of ground state. It is a big advantage arising from Andrews and Clutterbuck's work.

Now, recall the first section, we make an attempt to introduce the modulus of convexity to abstract Wiener space. It seems difficult to generalize the arguments of Andrews and Clutterbuck directly, due to the loss of compactness and regularity.

However, we still have similar results as follows.

# Notation

Denote by $(\mathbb{W}, \mathbb{H}, \mu)$ an abstract Wiener space and $\mathcal{L}_*$ the Ornstein–Uhlenbeck operator on $\mathbb{W}$.

Let $V \in \mathcal{D}_1^p(\mathbb{W}, \mu)$ for some $p > 1$ satisfy the KLMN condition (see Reed and Simon: *Methods of modern mathematical physics, IV*). Define

$$-\mathcal{L} = -\mathcal{L}_* + V$$

to be a self-adjoint Schrödinger operator bounded from below.

Correspondingly, denote by $\tilde{\mathcal{L}}_*$ the Ornstein–Uhlenbeck operator on $\mathbb{R}^1$ with respect to the Gaussian measure.

Let $\tilde{V} \in C^1(\mathbb{R}^1) \cap L^1(\mathbb{R}^1, \gamma_1)$ be a symmetric potential satisfying the KLMN condition too. Define

$$-\tilde{\mathcal{L}} = -\tilde{\mathcal{L}}_* + \tilde{V}.$$

# Variation formula

It is well known that, there are two equivalent min-max principles for any self-adjoint operator $H$ bounded from below.

That is $\mu_i = \lambda_i$ for all $i \geqslant 0$, which are defined as

1. $$\mu_i = \sup_{\varphi_0,\varphi_1,\ldots,\varphi_{i-1}} \inf_{\substack{\varphi\in\mathcal{D}[H],\|\varphi\|=1,\\ \varphi\in[\varphi_0,\varphi_1,\ldots,\varphi_i]^\perp}} (\varphi, H\varphi);$$
2. $$\lambda_i = \inf_{\varphi_0,\varphi_1,\ldots,\varphi_i\in\mathcal{D}[H]} \sup_{\substack{\|\varphi\|=1,\\ \varphi\in\mathrm{span}\{\varphi_0,\varphi_1,\ldots,\varphi_i\}}} (\varphi, H\varphi).$$

By convention, $\varphi_0, \varphi_1, \ldots, \varphi_i$ are all linearly independent and $[\varphi_0, \varphi_1, \ldots, \varphi_i]^\perp$ denotes the orthogonal completion of $\mathrm{span}\{\varphi_0, \varphi_1, \ldots, \varphi_i\}$.

# Main Theorems

**Theorem 13**

*Suppose for almost all $w \in \mathbb{W}$ and every $h \in \mathbb{H}$ with $h \neq 0$*

$$\left\langle \nabla V(w+h) - \nabla V(w), \frac{h}{\|h\|_{\mathbb{H}}} \right\rangle_{\mathbb{H}} \geqslant 2\tilde{V}'\left(\frac{\|h\|_{\mathbb{H}}}{2}\right). \tag{10}$$

*Then there exists a comparison*

$$\lambda_1 - \lambda_0 \geqslant \tilde{\lambda}_1 - \tilde{\lambda}_0.$$

Hence, the existence of the spectral gap of $-\mathcal{L}$ on Wiener space can sometimes be reduced to one dimensional case. Note that, $V$ doesn't need to be convex at all.

The next result gives the modulus of log-concavity for $\phi_0$.

### Theorem 14

*Assume the same condition as in Theorem 13 and the gap $\tilde{\lambda}_1 - \tilde{\lambda}_0 > 0$. Then $-\mathcal{L}$ and $-\tilde{\mathcal{L}}$ have a unique ground state respectively. Moreover, for almost all $w \in \mathbb{W}$ and every $h \in \mathbb{H}$ with $h \neq 0$,*

$$\left\langle \nabla \log \phi_0(w+h) - \nabla \log \phi_0(w), \frac{h}{\|h\|_{\mathbb{H}}} \right\rangle_{\mathbb{H}} \leqslant 2(\log \tilde{\phi}_0)' \left( \frac{\|h\|_{\mathbb{H}}}{2} \right).$$

Our proof relies on the approximation of eigenvalues and eigenfunctions, from bounded domains to $n$-dimensional Gaussian spaces and thus to Wiener space.

As a counterpart, we also compare $\lambda_1$ of diffusion operator

$$-\mathcal{L} = -\mathcal{L}_* + \nabla F \cdot \nabla$$

with $\tilde{\lambda}_1$ of the one dimensional operator

$$-\tilde{\mathcal{L}} = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} + (t + \omega'(t))\frac{\mathrm{d}}{\mathrm{d}t}.$$

Here, the two functions $F$ and $\omega$ are related by the following inequality: for all $h \in \mathbb{H}$ and $\mu$-a.e. $w \in \mathbb{W}$,

$$\left\langle \nabla F(w+h) - \nabla F(w), \frac{h}{\|h\|_{\mathbb{H}}} \right\rangle_{\mathbb{H}} \geqslant 2\omega'\left(\frac{\|h\|_{\mathbb{H}}}{2}\right).$$

**Theorem 15**

*Assume that $F \in \mathcal{D}_1^p(\mathbb{W}, \mathbb{R})$ satisfies $\int_{\mathbb{W}} \mathrm{e}^{-F}\,\mathrm{d}\mu = 1$. Suppose also that $\omega \in C^1(\mathbb{R})$ is even, satisfying $\int_{\mathbb{R}} \mathrm{e}^{-\omega}\,\mathrm{d}\gamma_1 = 1$ and $\lim_{t\to\infty}(\omega'(t) + t) = +\infty$. Then we have $\lambda_1 \geqslant \tilde{\lambda}_1$.*

# Outline

The following problems need to be studied in the further:

- How to prove the spectral gap comparison theorem by probabilistic method when $V$ has the non-zero modulus of convexity?

- How to prove spectral gap comparison theorem under Gong-Wu's condition on $b = \nabla V$ when $V$ has no global modulus of convexity?
- How to extend the spectral gap comparison theorem to path and loop spaces over compact Riemannian manifolds?

Thank you for your attention!